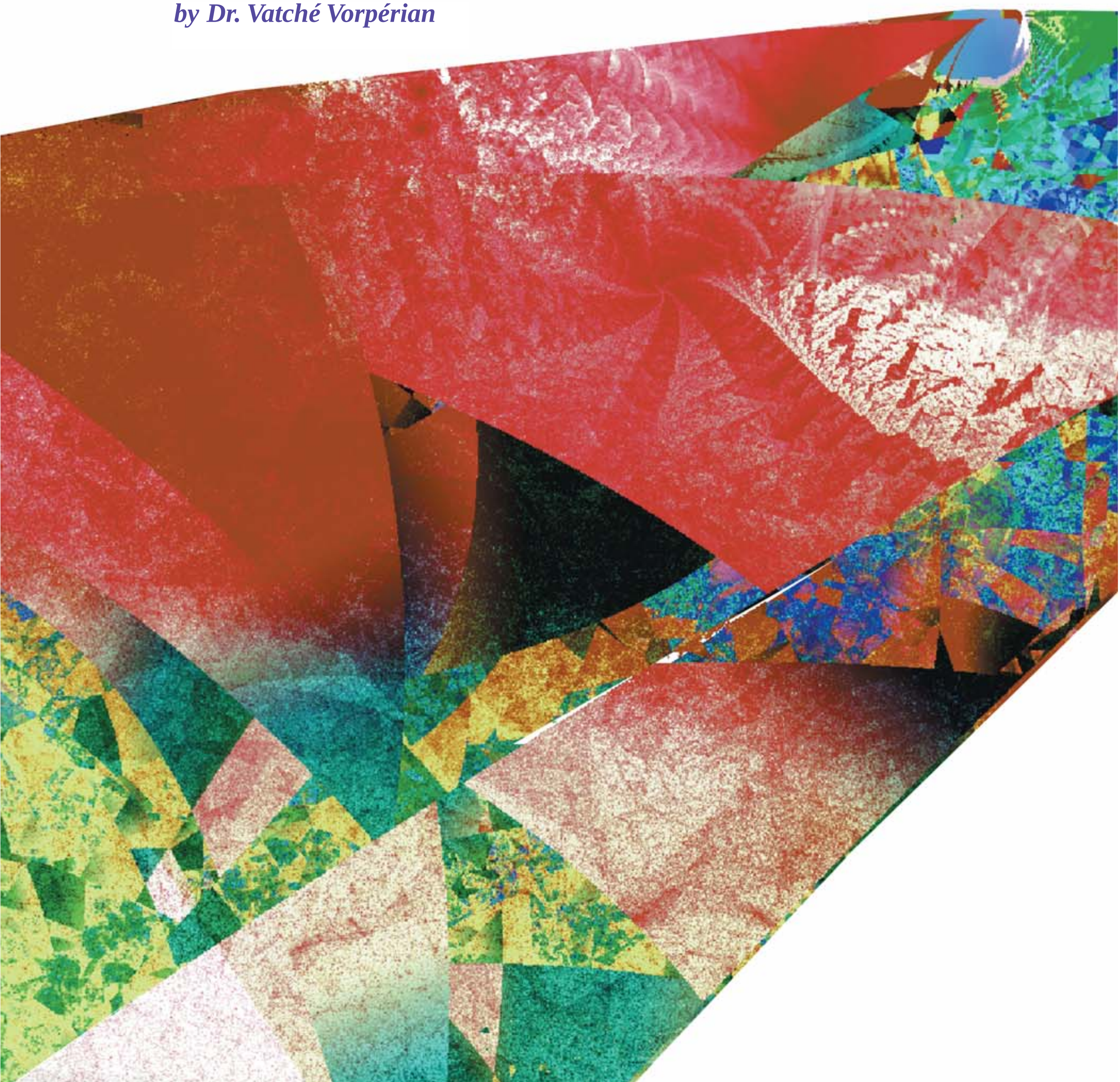
Using Fractals to Model Eddy-Current Losses in Ferromagnetic Materials

by Dr. Vatché Vorpérian



Losses in ferromagnetic materials are divided into hysteretic and eddy-current losses both of which arise from complicated processes inside the magnetic material. Although computation of eddy-current losses from first principles is extremely difficult, one can use concepts from fractal geometry to derive a circuit model that mimics their observed characteristic at the terminals of a transformer or an inductor. This article reviews one such circuit model [1], shown in Fig. 1, which emulates the correct dependence of eddy-current losses on the frequency of excitation.

Eddy-current losses are due to the finite conductivity of the magnetic material and have a non-integer power dependence on the frequency of excitation over the useful range of frequency of operation of the material. For high frequency ferrites, manufacturers provide total core loss curves and specify *K*, *a* and *b* in the following approximate core loss formula for different ferrites:

$$P_l = K B_m^a f^b \tag{1}$$

In this equation, B_m is the peak flux density and f is the frequency of excitation. The exponents *a* and *b* are non-integers so that *K* has very cumbersome and meaningless units. In this article, we will show a much better way of writing Eq. (1).

A physical law in which a quantity is raised to a non-integer power, such as the one in Eq. (1), is called a *scaling law* or a *power law* and is usually associated with some underlying fractal structure. An ideal fractal consists of an endless repetition of a scaling self-similar structures, so that one cannot stipulate an elemental volume *(dxdydz)* at any scale in which the structure is sufficiently smooth and possesses a certain density function (such as magnetic or electric flux density) that can be integrated over the entire volume of the structure. Hence, the calculus of fractal geometry is quite different from the calculus of analytic geometry, with which we are all familiar, in that it involves fractional derivatives, fractional integrals and scaling equations. In practice, a real material may behave like an ideal fractal only over a limited range of a certain parameter. For example, the circuit in Fig. 1 consists of a cascade of an infinite number of scaling *RL* branches, but only six or seven branches are needed to simulate eddy current losses.

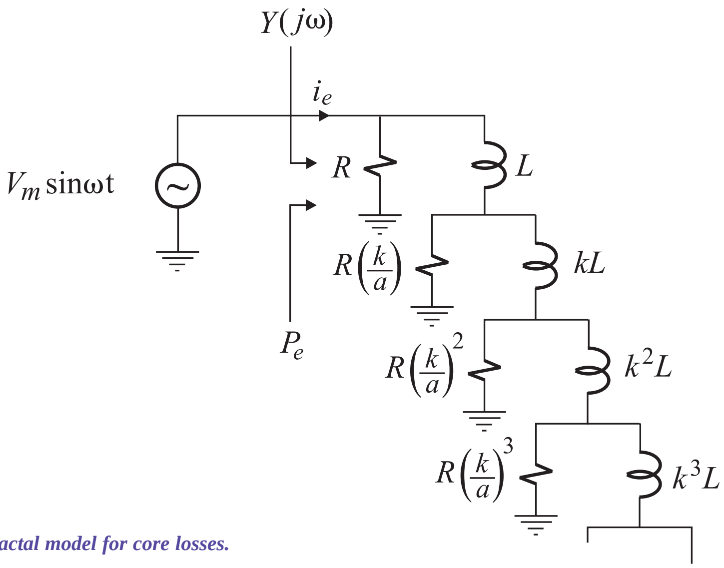


Fig. 1: Fractal model for core losses.

THE MODEL

The power loss in the network in Fig. 1, less the losses due to hysteresis, represents the losses due to eddy currents. Hence, we shall determine the admittance, $Y(j\omega)$, looking into the fractal network. At first, we shall assume that the network has an infinite number of cascades and demonstrates some elementary calculus of fractals. Later, we shall remove this restriction and determine the frequency range over which the results of the infinite network are valid.

The input admittance of the finite and infinite fractal network

The input admittance of the circuit in Fig. 1 can be written as a continued fraction

$$Y(j\omega) = G + \cfrac{1}{j\omega L + \cfrac{1}{G\dfrac{a}{k} + \cfrac{1}{j\omega kL + \cfrac{1}{G\left(\dfrac{a}{k}\right)^2 + \cfrac{1}{j\omega k^2 L + (\ldots)}}}}} \quad (2)$$

in which $G = 1/R$. Equation (2) can be rewritten by factoring out successive powers of a / k as follows:

$$Y(j\omega) = G + \cfrac{1}{j\omega L + \cfrac{k}{a}\cfrac{1}{G + \cfrac{1}{ja\omega L + \cfrac{k}{a}\cfrac{1}{G + \cfrac{1}{ja^2\omega L + (\ldots)}}}}} \qquad (3)$$

Equation (3) satisfies the following relation:

$$Y(j\omega) = G + \cfrac{1}{j\omega L + \cfrac{k}{aY(j\omega a)}} = G + \frac{Y(j\omega a)}{\frac{k}{a} + j\omega L Y(j\omega a)} \qquad (4)$$

An exact solution of Eq. (4) is difficult to obtain analytically so that we shall solve it approximately in the limits of high, medium and low frequencies. We can see from Fig. 1 and Eq. (4), that at high frequencies the input admittance simply becomes G:

$$\lim_{\omega \to \infty} Y(j\omega) \to G = \frac{1}{R} \qquad (5)$$

At low frequencies, we expect to have:

$$\lim_{\omega \to 0} \omega Y(j\omega) \to 0 \qquad (6)$$

so that Eq. (4) reduces to:

$$Y(j\omega) = \frac{a}{k} Y(ja\omega) \qquad (7)$$

Equation (7) is common to all simple geometric fractals (devil's staircase, Fournier universe, etc. [2]), modeled after the Cantor bar in which $a > k > 1$, and is known as a scaling equation. The solution of Eq. (7) is given by:

$$Y(j\omega) = K(j\omega)^{-\eta_f} \qquad (8)$$

in which K is an arbitrary constant and η_f is the celebrated fractal dimension given by:

$$\eta_f = \frac{\ln(a/k)}{\ln a} < 1 \qquad (9)$$

The constant K in Eq.(8) has the meaningless units of $\Omega^{-1}\sec^{\eta_f}$. A much better way to write this equation is to normalize ω with respect to ω_H , the point at which the magnitude of $Y(j\omega)$ in Eq. (8) is equal to G:

$$Y(j\omega) = G\left(\frac{\omega_H}{j\omega}\right)^{\eta_f} = G\left(\frac{\omega_H}{\omega}\right)^{\eta_f} e^{-j\frac{\pi}{2}\eta_f} \qquad (10)$$

Hence, ω_H is the frequency at which the two asymptotes of the magnitude response (Eqs. (5) and (10)) meet as shown in Fig. 2. Below ω_H , the slope of the magnitude asymptote is $20\eta_f$ dB/decade as can be ascertained from Eq. (2). It is rather easy to estimate ω_H by inspecting the circuit in Fig. 1 and realizing that ω_H is approximately the cut-off frequency of the first branch:

$$\omega_H = \frac{R}{L} \qquad (11)$$

A magnitude and phase plot using an actual circuit with only seven cascades is shown in Figs. 3a and b. The following circuit values were used:

$$k = 2,\ a = 10,\ L = 10\mu\text{H},\ R = 10^6\Omega,\ N = 7 \qquad (12)$$

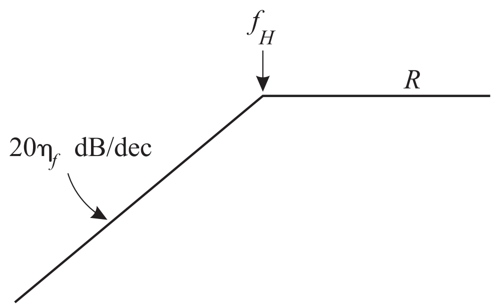


Figure 2:

Asymptotic magnitude plot of $Z(j\omega) = Y^{-1}(j\omega)$ of the infinite fractal network in Fig. 1.

The upper cutoff frequency ω_H and the fractal dimension η_f are computed to be:

$$\omega_H = \frac{R}{L} = 2\pi\,(15.9 \times 10^9)\ \text{rad/sec} \qquad (13a)$$

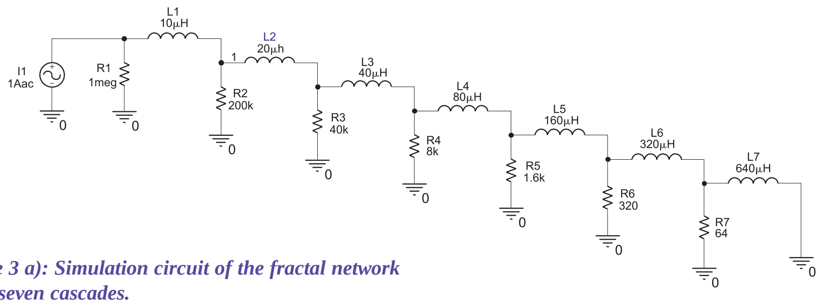


Figure 3 a): Simulation circuit of the fractal network using seven cascades.

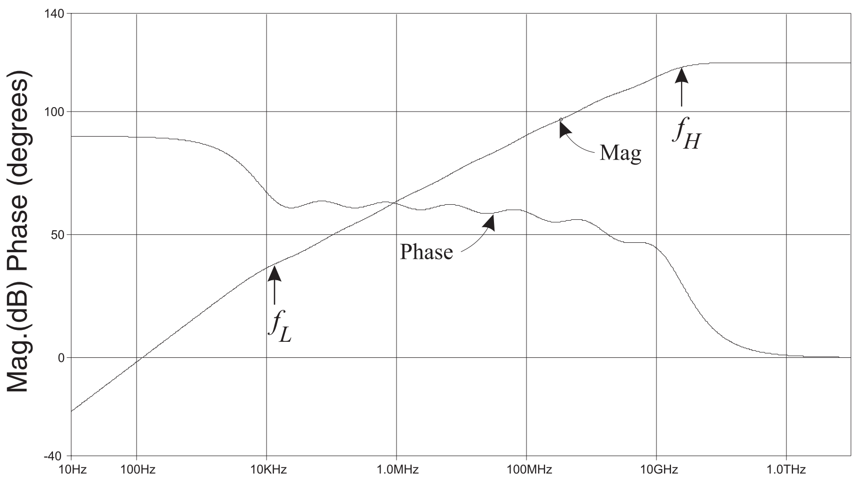


Figure 3 b): Magnitude and phase plot of $Z(j\omega) = Y^{-1}(j\omega)$.

$$\eta_f = \frac{\ln(10/2)}{\ln 10} = 0.69 \quad (13b)$$

The lower cutoff frequency is approximately given by the cutoff frequency of the last branch:

$$\omega_L = R\left(\frac{k}{a}\right)^N \frac{1}{Lk^N} = \frac{\omega_H}{a^N} \quad (14)$$

Below ω_L , the bounded network acts like an inductor consisting of the sum of all the inductors, $k^n L$,:

$$Y(j\omega) = \frac{1}{j\omega L \sum_{n=0}^{N} k^n} \quad ; \quad \omega < \omega_L \quad (15)$$

In fact, Eq. (15) is consistent with our expectation in that at very low frequencies, when the losses become negligible, we expect the admittance to be dominated by the inductance of the winding. Hence, it is reasonable that the summation in Eq. (15) should yield the inductance of the winding:

$$L_p = L \sum_{n=1}^{N} k^n = L \frac{k^{N+1} - 1}{k - 1} \quad (16)$$

The input admittance derived above for the bounded fractal network with N cascades is summarized as follows:

$$Y(j\omega) = \begin{cases} G & \omega > \omega_H \\ G\left(\dfrac{\omega_H}{j\omega}\right)^{\eta_f} & \omega_L < \omega < \omega_H \\ \dfrac{1}{j\omega L_p} & \omega < \omega_L \end{cases} \quad (17a\text{-}c)$$

Eddy current losses

The power dissipated in the fractal network represents the eddy-current losses which for sinusoidal excitations, $V_m \sin\omega t$, is given by:

$$P_e = \frac{V_m I_e}{2} \cos\phi \quad (18)$$

where ϕ is the phase angle between the voltage and current phasors. For the useful frequency range of operation, where the fractal behavior is dominant, the voltage and current phasors are related by the admittance function in Eq. (17b) so that Eq. (18) yields:

$$P_e = \frac{V_m^2}{2} G\left(\frac{f}{f_H}\right)^{-\eta_f} \cos\frac{\pi}{2}\eta_f \quad ; \quad f_L < f < f_H \quad (19)$$

The applied voltage is proportional to the peak flux density, B_m, and the frequency of excitation, f, so that we have:

$$V_m = AB_m f = AB_o f_H \frac{f}{f_H} \frac{B_m}{B_o} \quad (20)$$

in which A is a proportionality constant. Substitution of Eq. (20) in (19) yields:

$$P_e = \frac{(AB_o f_H)^2}{2} G \left(\frac{B_m}{B_o}\right)^2 \left(\frac{f}{f_H}\right)^{2-\eta_f} \cos\frac{\pi}{2}\eta_f \tag{21}$$

This equation correctly shows that eddy-current losses depend on a fractional power of the excitation frequency which is less than two and greater than one. In the same equation, the losses are seen to depend on the square of the peak flux density which, experimentally, is observed only in a few types ferromagnetic materials. We shall give a plausible model and an explanation for the more generally observed dependence of the eddy-current losses on a fractional power of the peak flux density. Since magnetic materials are nonlinear in general, their permeability generally depends on the peak flux density. Hence, we assume L_p, or L, depend on the peak flux density according to:

$$L = L_o \left(\frac{B_m}{B_o}\right)^{\eta_B} \tag{22}$$

in which η_B is referred to as the magnetic dimension and can vary quite widely. The upper cut-off frequency in Eq. (11) can now be written as:

$$f_H = \frac{1}{2\pi}\frac{R}{L_o}\left(\frac{B_m}{B_o}\right)^{-\eta_B} = f_o\left(\frac{B_m}{B_o}\right)^{-\eta_B} \tag{23}$$

Substituting Eq. (23) in (21) yields the desired loss formula:

$$P_e = P_{eo}\left(\frac{B_m}{B_o}\right)^{2-\eta_B\eta_f}\left(\frac{f}{f_o}\right)^{2-\eta_f} \tag{24}$$

in which:

$$P_{eo} = P_e(B_o, f_o) = \frac{(AB_o f_o)^2}{2} G \cos\eta_f \tag{25}$$

Equation (24) is a better way of writing losses than Eq. (1) simply because B_m and f are properly normalized and P_{eo} is simply the eddy current losses at a particular value of the peak flux density and frequency as shown in Eq. (25).

When the excitation frequency is increased beyond f_o, the input admittance asymptotes to G so that the losses are given by:

$$P_{eo} = \frac{V_m^2}{2} G \approx B_m^2 f^2 \quad ; \quad f > f_o \tag{26}$$

For most ferromagnetic materials, Eq. (26) is outside their practical range of operation because the cores get extremely hot unless they are operated at very low peak flux densities. It is customary to refer to the losses in Eq. (26) as classical eddy-current losses and those in Eq. (24) as anomalous eddy-current losses. It should be noted that if for a certain ferrite the normal frequency range of operation includes the classical range, $f > f_o$, then an additional resistor can be easily added in parallel with $R1$ in Fig. 3a to lower f_o to the desired value of the core. This important point is illustrated in Figs. 4a-d by way of the following example:

Consider a 1270 μH inductor wound on a core whose eddy current losses are modeled by the circuit in Fig. 3a in which f_o = 15GHz. When this inductor is driven by a square voltage waveform with 10V peak-to-peak and 50kHz, the resulting inductor current obtained from the simulation circuit in Fig. 3a is shown in Fig.4a and is seen having a slightly slanted triangular waveform because of the anomalous eddy current losses. For soft high frequency ferrites operating at moderate frequencies such a waveform is always seen. Note that the model in Fig. 3a does not have any hysteresis.

If we now increase the frequency to 500kHz, we see only the magnitude of the current waveform drop as shown in Fig. 4b because the core model assumes that the core is well suited for high frequencies since f_o = 15GHz. Now, let us assume that the core is not as superbly suited for high frequencies and has a moderate f_o = 1.5GHz, which can be easily modeled in Fig. 3a by adding a 5KΩ resistor in parallel with R1 (assuming the remaining characteristics are the same.) Now, when the inductor with the lossier core is driven with the same voltage waveform at 50kHz, there will hardly be any difference between its current, shown in Fig.4c, and that of the superior core, shown in Fig.4a. When the lossier core is now driven at 500kHz, a discontinuity begins to appear in the current waveform as shown in Fig.4d owing to the classical eddy current losses predicted by the fractal circuit model. This can be easily verified experimentally by pushing a soft ferrite to higher frequencies and observing the inductor current beginning to develop a discontinuity.

Physical interpretation of the model and experimental results

The fractal model in Fig. 1 and its power loss formula in Eq. (24) are so versatile that they can be made to fit just about any kind of experimental data— even bad data. For the proposed model to have any credibility, it should have a plausible physical interpretation, and be checked against eddy-current loss data that is obtained after a careful separation of hysteresis losses from total core losses. Such data has been reported in the literature [3], [4] and [5], against which a favorable verification of the proposed model is discussed in [1]. Another result reported in [6] describes a new alloy which has a constant permeability that does not vary with the flux density. According to Eq. (22), the magnetic dimension of this alloy should be zero, $\eta_B \approx 0$, so that the eddy current losses should depend on B^2 as required by Eq. (24). This turns out to be in excellent agreement with the measured loss data presented in [6].

Physically, the model suggests, in a very simplified manner, that the magnetization process and the eddy currents in the core are not uniformly distributed but are very localized. Each resistive branch in the model can be thought of as a localized eddy current path separated from the others by magnetized regions represented by inductors. Also, since each lossy eddy current path acts as a secondary load to the primary excitation, the inductors represent the leakage between the primary excitation and the eddy current paths. The non-uniformity of the magnetization process is a well-known fact and is ascribed to the dynamics of domain walls. It can be deduced from the results and the discussion in [3] that the input impedance of the fractal network in Fig. 1, given by the reciprocal of Eq. (10), is proportional to the number of bar-like domain walls participating in the flux reversal process. Since domain walls are known to get stuck for a while before they can continue to move again, it is reasonable to expect that their number should depend on the excitation frequency. Hence, the physical connection between the fractal network in Fig. 1 and the loss process in a ferromagnetic material seems to be related to the nature of domain wall motion. In amorphous materials, which do not posses bar-like domains, it is suggested in [3] that one can define equivalent bar-like domains.

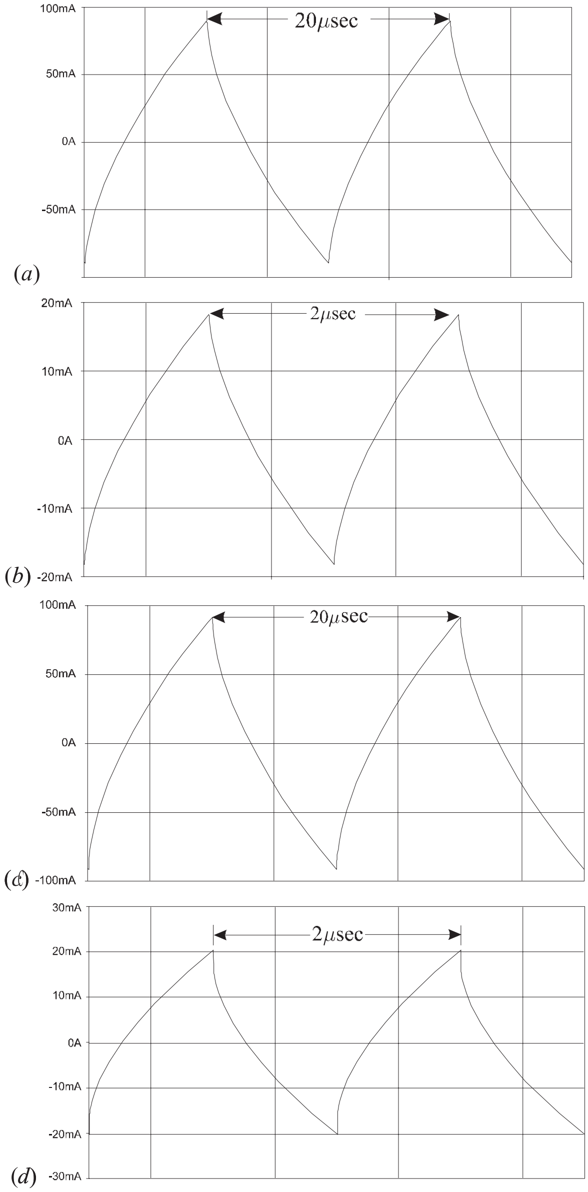


Figure 4 a) The current waveform of a 1270μH inductor with an eddy-current loss model given in Figs. 3a and b in which f_o = 15GHz. The inductor is driven by a 50kHz 10V square waveform.
b) The same current waveform in a) when the frequency is increased to 500kHz.
c) The same current waveform in a) when f_o = 1.5GHz in the eddy-current loss model simulated by adding 5KΩ in parallel with R1 in Fig. 3a.
d) The same current waveform in c) when the drive frequency is increased to 500kHz.
Note the jump in the current waveform due now to the classical eddy-currents.

The circuit model for eddy current losses presented here should be relatively easy to implement in any circuit simulation program. This model can be used to enhance existing non-linear hysteresis core models which use the Jiles-Atherton model [7].

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